Computers Math. Applic. Vol. 25, No. 10/11, pp. 79–88, 1993
0097-4943/93 $6.00 + 0.00

DISCONTINUITY AND MEASURABILITY OF ROBUST FUNCTIONS IN THE INTEGRAL GLOBAL MINIMIZATION

QUAN ZHENG
Department of Mathematics
Shanghai University of Science and Technology, Shanghai, China

Abstract—Robustness and measurability of a set and of a function are the foundation of the integral approach to global minimization. However, a robust function may be discontinuous and nonmeasurable. In this paper, we show that the set of points of discontinuity of a robust function has empty interior and is of the first category. A robust function on the interval [0,1] is constructed such that the Lebesgue measure of its set of points of discontinuity approaches 1. We also show that there is a robust set on [0,1] which is Lebesgue nonmeasurable, and then a Lebesgue nonmeasurable robust function is constructed.

1. INTRODUCTION

During a period at the end of the last century and in early 1900's, Baire [1], Borel [2], Lebesgue [3] and other scholars investigated discontinuities of a function on $\mathbb{R}^1$. The motivation of such studies was from the real analysis itself. With the development of the integral approach to global minimization, the study of different types of discontinuity becomes fundamental as concerns the *applicability* of the integral methods.

Let X be a topological space, D a subset of X and $f : X \to \mathbb{R}^1$ a real-valued function on X. Consider the following minimization problem:

$$c^* = \inf_{x \in D} f(x). \tag{1}$$

If f is measurable (with respect to a given Q-measure space) and robust on a robust set D, then one can use the integral method of global minimization to characterize the optimality and to find the set of robust global minimizers (see [3–9]). Here, the measurability and robustness of a set and of a function play an important role in the integral global minimization, providing the foundation of this approach.

We have comprehensive knowledge about measurable sets and functions. However, the study of robust sets and functions has just started. A measurable set, usually, is nonrobust; neither is a measurable function. It is natural to ask if a robust set or a robust function is measurable. A robust function may be discontinuous; but what about the structure of the set of points of discontinuity of a robust function?

After recalling basic definitions and properties of robust sets and functions in Section 2, we present a further study of robust sets and functions. In Section 3, the classification of discontinuities of a bi-robust function defined on a topological space is explored, which generalizes the concept of discontinuities of the first kind and second kind of a function defined on the real line $\mathbb{R}^1$. This can be made by using singularity of points of a robust set. In Section 4, we prove that the interior of the set D_r of points of discontinuity of a robust function is empty; it follows that D_r is of the first category, and the set of points of continuity is of the second category. However, the measure of D_r may not be small. In Section 5, we first construct an example on the interval [0,1] that the Lebesgue measure of D_r approaches 1. Then, an example of a Lebesgue nonmeasurable robust set on [0,1] is constructed. This allows us to prove that there is a robust function on [0,1] which is Lebesgue nonmeasurable.

CAMWA 25/10-11-G

2. ROBUST SETS AND FUNCTIONS

Let X be a topological space, D a subspace of X. If

$$\text{cl } D = \text{cl int } D, \tag{2}$$

then D is called robust set in X; here int D denotes the interior of D and cl D the closure of D.

An open set in X, including X itself and the empty set $\emptyset$, is robust. A closed set may be robust or nonrobust. A union of robust sets is robust. However, an intersection of two robust sets may be nonrobust; the intersection of a robust set and an open set is robust. The interior of a robust set is nonempty.

Note that the concept of robustness is closely related to the kind of topology endowed with X. For example, the set

$$U_M = \{u(t) \mid \ |u(t)| \le M,\, u \in L_2[0,1]\} \tag{3}$$

is nonrobust in $L_2[0,1]$ since int $U_M = \emptyset$. However, U_M is robust with respect to relative topology on it, see [10,11].

A point $x \in \text{cl } D$ is said to be a robust point to D if, for each neighborhood $N(x)$ of x,

$$N(x) \cap \text{int } D \neq \emptyset. \tag{4}$$

If, moreover, $x \in D$, then x is said to be a robust point of D.

We can use a sequencial terminology to characterize a robust point in a metric space.

Proposition 2.1. *Let (X,d) be a metric space and D a subset of X. A point x is robust to D if and only if there is a sequence $\{x_k\} \subset$ int D such that $x_k \to x$ as $k \to \infty$.*

A set D is robust if and only if each point of D is a robust point of D.

Now, let D be a nonempty robust set in a topological space. Then int $D \neq \emptyset$. An open set in a topological space can be decomposed into a union of disjoint connected open sets:

$$\text{int } D = \bigcup_{\alpha \in \Lambda} G_\alpha,$$

where G_α, $\alpha \in \Lambda$ are connected open sets and $G_{\alpha_i} \cap G_{\alpha_j} = \emptyset$, $\forall\, \alpha_i \neq \alpha_j$ (here, indices i, j are used to distinguish between different α's and they do not mean that the union is finite or countable). With this decomposition, we can classify robustness of points as follows.

Definition 2.1. *A point x is said to be regularly robust to a set D in a topological space X if x is robust to D and there are a neighborhood $N(x)$, an integer n, and indices $\alpha_1, \dots, \alpha_n$ in Λ, such that*

$$N(x) \cap G_\alpha = \emptyset, \qquad \forall\, \alpha \neq \alpha_1, \dots, \alpha_n. \tag{5}$$

A point x is said to be singularly robust to D, if it is robust to D and not regularly robust to it. That is, if x is singularly robust to D, then each neighborhood $N(x)$ of x has nonempty intersection with infinitely many connected components of int D. *If, moreover, D is a subset of a separable metric space, then each open set can be decomposed into a union of a countable number of connected open sets, and a point x is sigularly robust to D if and only if there is a sequence $\{x_k\}$, such that $x_k \to x$ as $k \to \infty$, which is contained in infinitely many connected components of int D.*

The structure of an open connected set in $\mathbb{R}^1$ is quite simple; it is just an open interval (a,b). Thus, the open set int D on $\mathbb{R}^1$ can be decomposed into a union of a countable number of open intervals. A point x, which is robust to a set D, is regular if there is a neighborhood $(x-\delta, x+\delta)$ such that it intersects at most two component intervals of int D.

Proposition 2.2. *Let D be a robust set on $\mathbb{R}^1$. A point $x \in$ cl D is singularly robust to D if and only if there is a sequence of component intervals $\{(a_n, b_n)\}$ of* int D *such that $a_n \to x$ and $b_n \to x$ as $n \to \infty$.*

Remark 2.1. Suppose x is singularly robust to D on $\mathbb{R}^1$; then there is an infinite number of $\{(a_n, b_n)\}$ on one side of the point x with the property mentioned in Proposition 2.2. Thus, we can take the sequence in such a way that $b_n > a_n > b_{n+1} > a_{n+1} > x$, $n = 1, 2, \ldots$, (in the case that there is an infinite number of intervals of the sequence on the right hand side of the point x) or $a_n < b_n < a_{n+1} < b_{n+1} < x$, $n = 1, 2, \ldots$, (in the left side case). Moreover, we have $[b_n, a_{n+1}] \cap D^c \neq \emptyset$, $n = 1, 2, \ldots$, (in the left side case) or $[b_{n+1}, a_n] \cap D^c \neq \emptyset$, $n = 1, 2, \ldots$, (in the right side case); here $D^c = \mathbb{R}^1 \setminus D$, the complement of D in $\mathbb{R}^1$.

REMARK 2.2. Suppose x is singularly robust to a set D on $\mathbb{R}^1$ and there is a sequence of components $\{(a_n, b_n)\}$ of int D on the *left* side of the point x, such that $a_n \to x$ and $b_n \to x$ as $\to \infty$. Then the set $(-\infty, x] \cap D$ is robust. Indeed, each point of $(-\infty, x) \cap D$ is a robust point of the set. We need only to prove that the point x is robust to the set D. Take a sequence of points $y_n \in (a_n, b_n)$, $n = 1, \ldots$ These points are in $(-\infty, x) \cap \operatorname{int} D = \operatorname{int}((-\infty, x) \cap D)$, and $y_n \to x$. It follows by Proposition 2.1 that x is robust to D. In this case, we say that x is singularly robust to D on the left.

REMARK 2.3. If x is singularly robust to a family of sets D_α on the real line $\mathbb{R}^1$ and the number of sets in the family is greater than three, then the point is singularly robust to more than one set on one side (on the left or right side).

EXAMPLE 2.1. The union of open intervals $D_0 = \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})$ is open; so, it is robust. Each point of D_0 is a regular robust point. Let $D = D_0 \cup \{0\}$, then, it is also robust. The point 0 is a singular robust point of D.

The existence of singular robustness makes the structure of a robust set more complicated. We will examine singularity in robust analysis further in the next sections.

A real-valued function f on X is said to be robust on a robust subset D if for each real number c, the set

$$F_c = \{x \in D \mid f(x) < c\} \tag{6}$$

is robust. An upper semicontinuous function on D is robust. Indeed, for each c, the set $F_c = D \cap \{x \mid f(x) < c\}$ is a robust set since it is an intersection of a robust set and an open set. A probabilistic distribution function on $\mathbb{R}^n$ is also an example of robust function. If f_α, $\alpha \in \Lambda$ is a family of robust functions, then $\inf_{\alpha \in \Lambda} f_\alpha(x)$ is still robust.

Note that we introduced the concept of robust function for a minimization problem. If one considers a maximization problem, it requires $-f$ to be a robust function. Or, it is equivalent that for each real number c the set

$$G_c = \{x \in D \mid f(x) > c\} \tag{7}$$

is robust. For a function f, if both f and $-f$ are robust, then it is called a *bi-robust function*.

A function f is said to be robust at (or, by) a point $x \in X$ if $x \in F_c$ implies that x is a robust point of F_c (or, if there is a neighborhood $N(x)$ of x such that $N(x) \cap F_c$ is a robust set). A fuction f is robust if and only if f is robust at (or, by) each and every point of X.

PROPOSITION 2.3. *Suppose D is a robust set in a topological space X. Then the function*

$$f(x) = \begin{cases} 1, & \text{if } x \in X \setminus D, \\ 0, & \text{if } x \in D, \end{cases}$$

is robust on X.

PROOF. For each real number c, we have

$$F_c = \{x \in X \mid f(x) < c\} = \begin{cases} X, & \text{if } 1 < c < \infty, \\ D, & \text{if } 0 < c \leq 1, \\ \emptyset, & c \leq 0. \end{cases}$$

Since X, D and $\emptyset$ are robust, so f is a robust function.

Suppose f is a bounded robust function on $X, |f(x)| \leq M$, $\forall x \in X$, where M is a constant. Divide the interval $[-M, M]$ equally into $n = 2^k$ subintervals, and let the points of division be

$$-M = c_0 < c_1 < \cdots < c_n = M.$$

Let, for a given i,

$$f_i(x) = \begin{cases} M, & \text{if } x \in X \setminus F_{c_i}, \\ c_i, & \text{if } x \in F_{c_i}. \end{cases}$$

Since the set $F_{c_i} = \{x \in X \mid f(x) < c_i\}$ is robust, we can prove, similarly to Proposition 2.3, that f_i is a robust function on X. Let

$$g_n(x) = \min\{f_1(x), f_2(x), \dots, f_n(x)\}.$$

Then g_n is also a robust function on X. It has constant values c_i on each set $F_{c_i} \setminus F_{c_{i-1}}$, $i = 1, 2, \dots, n$, i.e., g_n is a robust "step" function and takes only n different function values. For each point $x \in X$, there are c_i and c_{i+1} such that $c_i \le f(x) < c_{i+1}$. It implies that $x \in F_{c_{i+1}}$, but $x \notin F_{c_i}$. Thus, $x \in F_{c_{i+1}} \setminus F_{c_i}$, and then $g_n(x) = c_{i+1}$. Hence,

$$|g_n(x) - f(x)| \le c_{i+1} - c_i = \frac{1}{2^k}.$$

Therefore, as $n = 2^k \to \infty$, $g_n(x)$ converges to $f(x)$ uniformly. We then have proved the following proposition.

PROPOSITION 2.4. *A bounded robust function on a topological space is the limit of a uniformly convergent sequence of robust step functions.*

3. CLASSIFICATION OF DISCONTINUITIES

In analysis, discontinuities of a function on $\mathbb{R}^1$ are classified as those of first and second kind; but this leaves no idea about how to classify discontinuities for a function in $\mathbb{R}^n$. For instance, the function

$$f(x, y) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \le 0, \end{cases} \tag{8}$$

is discontinuous on the line $x = 0$ in $\mathbb{R}^2$. One can accept the idea that f has a discontinuity of first kind at each point on this line. However, it is easy to find a sequence $\{(x_n, y_n)\}$ converging to, say, the point (0,0), and such that the limit of the sequence $\{f(x_n, y_n)\}$ does not exist. Utilizing the classification of robust points of a robust set, we shall try to classify discontinuities of a bi-robust function. We start with proving a characterization of discontinuities of a bi-robust function on the real line.

THEOREM 3.1. *Suppose that f is a bi-robust function on $\mathbb{R}^1$. Then f has a discontinuity of second kind at a point x_0 if and only if there is a real number c and a positive number η, such that $x_0 \in F_{c-\epsilon}$ (or $G_{c+\epsilon}$) and the point x_0 is a singular robust point of $F_{c-\epsilon}$ (or $G_{c+\epsilon}$) for all $0 \le \epsilon < \eta$.*

PROOF. *Sufficiency.* Suppose the point x_0 is a singular robust point of F_c. Let $\{I_n = (a_n, b_n),\ n = 1, 2, \dots\}$ be a sequence of component intervals within int F_c such that $a_n \to x_0$ and $b_n \to x_0$ as $n \to \infty$, and $[b_n, a_{n+1}] \cap (F_c)^c \ne \emptyset$ (we consider the case that there is a sequence of components of int F_c on the left hand side of the point x_0, see Remarks 2.1 and 2.2, i.e., we can assume that $a_n < x_0$ and $b_n < x_0$ for $n = 1, 2, \dots$ Take points $z_n \in [b_n, a_{n+1}]$ such that $z_n \in (F_c)^c$, $n = 1, 2, \dots$ We have $z_n \to x_0$ and $f(z_n) \ge c$, $n = 1, 2, \dots$ Hence, $\limsup_{x \to x_0 - 0} f(x) \ge c$.

On the other hand, $x_0 \in F_c$ implies $f(x_0) < c$. Let $\epsilon = \min(\frac{\eta}{2}, \frac{1}{2}[c - f(x_0)]) > 0$. We still have $x_0 \in F_{c-\epsilon}$, and x_0 is a robust point of $F_{c-\epsilon}$ either. Without loss of generality, we can take $\epsilon > 0$ such that $(-\infty, x_0) \cap \text{int } F_{c-\epsilon} \ne \emptyset$ (see Remarks 2.2 and 2.3). Thus, for $\delta_n \downarrow 0$, $(x_0 - \delta_n, x_0] \cap \text{int } F_{c-\epsilon} \ne \emptyset$. Taking $y_n \in (x_0 - \delta_n, x_0]$, we have $f(y_n) < c - \epsilon$ and $y_n \to x_0$. Thus, $\liminf\limits_{x \to x_0 - 0} f(x) \le c - \epsilon$. We have proved

$$\liminf_{x \to x_0 - 0} f(x) \le c - \epsilon < c \le \limsup_{x \to x_0 - 0} f(x). \tag{9}$$

The left limit of f at x_0 does not exist. Therefore, f has a discontinuity of the second kind at the point x_0.

Necessity. Since f has a discontinuity of the second kind at x_0, one of the following inequalities holds:

$$\liminf_{x\to x_0-0} f(x) < \limsup_{x\to x_0-0} f(x) \quad \text{or} \quad \liminf_{x\to x_0+0} f(x) < \limsup_{x\to x_0+0} f(x). \tag{10}$$

Say, $\liminf_{x\to x_0-0} f(x) = \alpha < \beta = \limsup_{x\to x_0-0} f(x)$. Take $\frac{1}{2}\eta = (\beta-\alpha)$ and a real number c such that $\alpha < c-\eta < c+\eta < \beta$. Let $c_1 \in (c-\eta, c+\eta)$ and $c_1 \neq f(x_0)$. Thus, either $x_0 \in F_{c_1}$, or $x_0 \in G_{c_1}$. Say, $x_0 \in F_{c_1}$.

We now prove that the point x_0 is singularly robust to F_{c_1}. First, we prove that for each $\delta > 0$, $(x_0-\delta, x_0) \cap \operatorname{int} F_{c_1} \neq \emptyset$. Indeed, since $\liminf_{x\to x_0-0} f(x) = \alpha$, there is a sequence $\{x_k\}$, $x_k < x_0$, $x_k \to x_0$ and $f(x_k) \to \alpha$, as $k \to \infty$. Thus, $x_k \in (x_0-\delta, x_0)$ and $f(x_k) < c+\eta < c_1$, when k is large enough. These mean that $(x_0-\delta, x_0)$ is a neighborhood of x_k and $x_k \in F_{c_1}$. The robustness of f implies that $(x_0-\delta, x_0) \cap \operatorname{int} F_{c_1} \neq \emptyset$.

We next prove that $x_0 \notin \operatorname{int} F_{c_1}$. Suppose $x_0 \in \operatorname{int} F_{c_1}$, then there is a neighborhood $N(x_0)$ of x_0 such that $N(x_0) \subset \operatorname{int} F_{c_1}$. However, from $\limsup_{x\to x_0-0} f(x) = \beta > c+\eta$, one can find a sequence which converges to x_0 and the function values at these points are greater than c_1, thus, $x_0 \notin \operatorname{int} F_{c_1}$. Let $(-\infty, x_0) \cap \operatorname{int} F_{c_1} = \bigcup_{i=1}^{\infty} I_i$.

Now suppose, on the contrary, that x_0 is a regular robust point. Then, when δ is small enough, the interval $(x_0-\delta, x_0)$, eventually, can only have a nonempty intersection with only one open interval of $\{I_i\}$, say, it is $I_{i_0} = (a, b)$. The right end point b cannot be greater than x_0 since $(a, b) \subset (-\infty, x_0)$; b cannot be smaller than x_0 since the intersection $(x_0-\delta, x_0) \cap \operatorname{int} F_{c_1}$ would then be empty when δ is small enough. Thus, $b = x_0$, i.e., we have an interval $(a, x_0) \subset \operatorname{int} F_{c_1}$. It follows from $\limsup_{x\to x_0-0} f(x) = \beta$ that there exists a sequence $\{y_n\}$ such that $y_n < x_0$, $y_n \to x_0$, and $f(y_k) \geq c+\eta$, for $k \geq n_1$, where n_1 is a large positive integer. However, $y_k \in (x_0-\delta, x_0)$, so $y_k \in \operatorname{int} F_{c_1} \subset F_{c_1}$ when k is large enough. This contradicts $f(y_k) > c+\eta$. Therefore, x_0 is a singular robust point of F_{c_1}. ∎

COROLLARY. *Suppose f is bi-robust on $\mathbb{R}^1$. If for each real number c, each point of sets F_c and G_c is a regular robust point, then f is continuous or has a discontinuity of first kind.*

Remark 3.1. It is possible that even though for some real number c, $x_0 \in F_c$ and x_0 is a singular robust point of F_c, this point x_0 is still a point of discontinuity of first kind of f. Let

$$f(x) = \begin{cases} x, & x \in (-\infty, 1) \bigcup \left\{ \cup_{n=1}^{\infty} [-\frac{1}{2n}, -\frac{1}{2n+1}] \right\}, \\ -x, & x \in \bigcup_{n=1}^{\infty} \left(-\frac{1}{2n+1}, -\frac{1}{2n+2} \right), \\ x-1, & x \in [0, \infty). \end{cases}$$

The function is bi-robust. Each point of $\mathbb{R}^1$ is a point of continuity or of discontinuity of first kind. Take $x_0 = 0$ and $c = 0$. Since $f(0) = -1 < c = 0$, so $x_0 \in F_c$. However,

$$F_c = [0, \infty) \cup \bigcup_{n=1}^{\infty} \left(-\frac{1}{2n+1}, -\frac{1}{2n+2} \right).$$

The point $x_0 = 0$ is singularly robust to F_c.

Motivated by the above results, we now classify discontinuities of a bi-robust function as follows.

DEFINITION 3.1. Suppose f is a bi-robust function on a topological space X and discontinuous at a point $x_0 \in X$. If there are a real number c and a positive number η such that $x_0 \in F_{c-\epsilon}$ (or $G_{c+\epsilon}$), and x_0 is a singular robust point of $F_{c-\epsilon}$ (or $G_{c+\epsilon}$) for all $0 \leq \epsilon < \eta$ then x_0 is said to be a point of discontinuity of the second kind. If a point of discontinuity of f is not of the second kind, then it is a point of discontinuity of the first kind.

With the above definition, each point of discontinuity of the function (8) is of the first kind.

4. SETS OF CONTINUITY AND DISCONTINUITY OF A ROBUST FUNCTION

Suppose f is a real function on $\mathbb{R}^1$, then the set of points of discontinuity of f may be $\mathbb{R}^1$ itself, or a set dense in $\mathbb{R}^1$, or a finite number of isolated points. However, if f is a robust function, then the set of points of discontinuity (or continuity) of f has a special structure. The following theorem tells us that the set of points of discontinuity of a robust function on a complete metric space has an empty interior.

Theorem 4.1. Suppose (X,d) is a complete metric space, f a bounded robust function on X, and D_r the set of points of discontinuity of f. Then

$$\text{int } D_r = \emptyset. \tag{11}$$

PROOF. Suppose, on the contrary, that int $D_r \neq \emptyset$; then there is a nonempty ball $B_1 = \{x \mid d(x,x_1) < r_1\}$ such that $B_1 \subset \text{int } D_r$. Let $c_1^* = \inf_{x\in B_1} f(x)$, then c_1^* is finite. Take a real number $c_1 > c_1^*$, then the set $F_{c_1} \triangleq B_1 \cap \{x \in X \mid f(x) < c_1\}$ is robust, since it is an intersection of an open set and a robust set. Moreover, by the definition of c_1^* and $c_1 > c_1^*$, we also have $F_{c_1} \neq \emptyset$. Thus, int $F_{c_1} \neq \emptyset$, and there is a nonempty ball $B_2 = \{x \mid d(x,x_2) < r_2\}$ such that $B_2 \subset \text{int } F_{c_1} \subset F_{c_1} \subset B_1$. The ball B_2 can be taken such that $\bar{B}_2 = \{x \mid d(x,x_2) \leq r_2\} \subset \text{int } F_{c_1} \subset B_1$. Let $c_2^* = \inf_{x\in B_2} f(x)$, then $c_1^* \leq c_2^* < c_1$. Take $c_2 = \frac{1}{2}(c_1 + c_2^*)$ and let $F_{c_2} \triangleq B_2 \cap \{x \in X \mid f(x) < c_2\}$. Since $c_2 > c_2^*$, so F_{c_2} is a nonempty robust set and int $F_{c_2} \neq \emptyset$, etc. In general, from int $F_{c_k} \neq \emptyset$, we can take a nonempty ball $B_{k+1} = \{x \mid d(x,x_{k+1}) < r_{k+1}\}$ such that $B_{k+1} \subset \text{int } F_{c_k}$ and $\bar{B}_{k+1} = \{x \mid d(x,x_{k+1}) \leq r_{k+1}\} \subset B_k$. Let

$$c_{k+1}^* = \inf_{x\in B_{k+1}} f(x). \tag{12}$$

Since $B_{k+1} \subset B_k$, so $c_k^* \leq c_{k+1}^*$. Moreover, for each $x \in B_{k+1} \subset \text{int } F_{c_k} \subset F_{c_k}$, we have $f(x) < c_k$, thus, $c_{k+1}^* < c_k$. Take

$$c_{k+1} = \frac{1}{2}(c_k + c_{k+1}^*), \qquad (< c_k). \tag{13}$$

With the above procedure, we obtain a decreasing sequence $\{c_k\}$ and an increasing sequence $\{c_k^*\}$. Furthermore, we have,

$$\begin{aligned} c_{k+1} - c_{k+2}^* &= \frac{1}{2}(c_k + c_{k+1}^*) - c_{k+2}^* \\ &\leq \frac{1}{2}(c_k + c_{k+1}^*) - c_{k+1}^* = \frac{1}{2}(c_k - c_{k+1}^*) \\ &\leq \cdots \leq \frac{1}{2^k}(c_1 - c_2^*) \to 0, \qquad \text{as } k \to \infty. \end{aligned}$$

Thus, the limits of both of $\{c_k\}$ and $\{c_k^*\}$ exist and are equal.

$$\lim_{k\to\infty} c_k^* = \bar{c} = \lim_{k\to\infty} c_k. \tag{14}$$

As a by-product, we also obtain a sequence of balls, $\{B_k\}$, such that

$$\bar{B}_{k+1} \subset B_k, \qquad k = 1,2,\ldots \tag{15}$$

On the ball B_{k+1}, we have

$$c_{k+1}^* \leq f(x) < c_k. \tag{16}$$

Let

$$\bar{B} = \bigcap_{k=1}^{\infty} \bar{B}_k,$$

then, from completeness of the space X, we have $\bar{B} \neq \emptyset$ (see [12,13]). Take a point $\bar{x} \in \bar{B} \subset \text{int } D_r$, thus,

$$c_{k+1}^* \leq f(\bar{x}) < c_k, \qquad k = 1,2,\ldots$$

Letting $k \to \infty$, we obtain that

$$f(\bar{x}) = \bar{c}.$$

We are now going to prove that f would be continuous at the point $\bar{x}$. Let $\{x_n\}$ be a sequence converging to $\bar{x}$. For each k, we have $\bar{x} \in B_k$, thus, there is an integer N such that $x_n \in B_{k+1}$ as $n > N$. It implies, from (16), that

$$c^*_{k+1} \leq f(x_n) < c_k, \qquad \forall n > N.$$

Letting $n \to \infty$ first, we take upper and lower limits,

$$c^*_{k+1} \leq \liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n) \leq c_k.$$

We then let $k \to \infty$ and obtain, from (14),

$$\bar{c} \leq \liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n) \leq \bar{c}.$$

These imply that the limit of the sequence $\{f(x_n)\}$ exists and

$$\lim_{n\to\infty} f(x_n) = \bar{c} = f(\bar{x})$$

for each given sequence $\{x_n\}$ converging to $\bar{x}$, i.e., f is continuous at $\bar{x}$. However, $\bar{x} \in \bar{B} \subset B_1 \subset \text{int } D_r \subset D_r$, which contradicts the definition of D_r. Hence, we have proved that int $D_r = \emptyset$. ∎

REMARK 4.1. We know that the Dirichlet function

$$D(x) = \begin{cases} 1, & \text{if } x \text{ is irrational,} \\ 0, & \text{if } x \text{ is rational,} \end{cases}$$

is discontinuous everywhere. Theorem 4.1 tells us that this situation would not happen for a robust function.

The following theorem confirms that the set D_r of points of discontinuity of a robust function is, moreover, of first category. Recall that a set in a metric space X is said to be of first category if it can be represented as a countable union of nowhere dense sets. A subset of X that cannot be so represented is said to be of second category.

THEOREM 4.2. *Suppose f is a bounded robust function on a complete metric space; then the set D_r of its points of discontinuity is of first category.*

PROOF. We know that the set D_r of points of discontinuity of a function f is a F_σ-type set, i.e., there is a sequence $\{F_n\}$ of closed sets such that

$$D_r = \bigcup_{n=1}^{\infty} F_n,$$

see [4, p. 31] From Theorem 4.1, we have int $D_r = \emptyset$. Thus, int $F_n \subset$ int $D_r = \emptyset, n = 1, 2, \ldots$ It implies that cl int $F_n = \emptyset, n = 1, 2, \ldots$, i.e, each set F_n is a nowhere dense set. Hence, we have proved that D_r is a union of countable nowhere dense sets; it is, then, of first category. ∎

The set of points of discontinuity of a function f is of first category if and only if f is continuous at a dense set of points (see [4]); thus, we have the following corollary.

COROLLARY 1. *Suppose f is a bounded robust function defined on a complete metric space X; then the set C_r of points of continuity of f is dense in X.*

REMARK 4.2. The set of points of continuity of a function may consist of a finite number of isolated points in a finite interval. For example, let

$$f(x) = \begin{cases} |x|, & \text{if } x \text{ is irrational,} \\ 0, & \text{if } x \text{ is rational.} \end{cases}$$

This function has a unique point $x = 0$ of continuity which is isolated. From Corollary 1, this never happens for a robust function.

COROLLARY 2. *The set of points of continuity of a bounded robust function on a complete metric space is of second category.*

REMARK 4.3. We cannot imply from the above theorems that the interior of the set of points of continuity of a bounded robust function is nonempty. In fact, the interiors of both sets of points of continuity and discontinuity may be empty. Let $X = (0,1)$ and consider the Riemann function:

$$R(x) = \begin{cases} \frac{1}{p}, & \text{if } x = \frac{q}{p} \text{ is a rational number in the irreducible form,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

We know that the set of points of continuity of R is the set of irrational numbers and that of discontinuity is the set of rational numbers in (0,1). Both of them are dense in X and their interiors are empty. It remains to prove that the Riemann function is robust. Indeed, for each given $c > 0$, the set $F_c = \{x \in (0,1) \mid f(x) < c\}$ consists of all of the points in (0,1) except a finite number of rational points where $1/p \geq c$. Indeed, for a fixed $c > 0$, the number of positive integers p such that $p \leq 1/c$ is finite; and for a fixed integer p, the number of rational numbers in the irreducible form $q/p \in (0,1)$ is less than p. Thus, the set F_c is a union of a finite number of intervals and it is robust. If $c \leq 0$, then F_c is an empty set, which is also robust. This proves that the Riemann function $R(x)$ is robust.

5. MEASURABILITY OF ROBUST SETS AND FUNCTIONS

In the last section we have proved that the set of points of discontinuity of a robust function is of the first category, and that of continuity is of the second category. It seems, at first glance, that D_r is "smaller" than C_r. In this section, we first give an example of a robust function on [0,1] for which the Lebesgue measure of D_r approaches 1.

To do this, we introduce two kind of robust sets on the real line $\mathbb{R}^1$. Suppose that $a < b$ and let

$$B_0(a,b) = \bigcup_{k=1}^{\infty} \left(a + \frac{b-a}{k+1}, a + \frac{b-a}{k}\right) \tag{17}$$

and

$$B_1(a,b) = \bigcup_{k=1}^{\infty} \left(b - \frac{b-a}{k}, b - \frac{b-a}{k+1}\right). \tag{18}$$

The sets $B_0(a,b)$ and $B_1(a,b)$ are unions of open intervals; they are robust on $\mathbb{R}^1$. Let $D_0 = B_0(a,b) \cup \{a\}$ and $D_1(a,b) = B_1(a,b) \cup \{b\}$, then D_0 and D_1 are also robust. Indeed, int $D_0 = B_0(a,b)$ and int $D_1 = B_1(a,b)$. The intersection of int D_0 and each neighborhood $(a-\delta, a+\delta)$ of the point a is nonempty. Thus, a is a robust point of D_0, so D_0 is robust. Similarly, D_1 is robust. The points a and b are singular robust points of D_0 and D_1, respectively. Utilizing such B_0-B_1 type sets, we can construct robust sets and functions with unusual properties.

Let α be given, $0 \leq \alpha < 1$; we construct a robust set on [0,1] such that the Lebesgue measure of its set of singular robust points is α and α can be close to 1.

Let $A_0 = [0,1]$ and $\beta = (1-\alpha)/(3-2\alpha)$. Let A_1 be the set obtained from A_0 by deleting an interval centered at the middle of A_0 with a lenghth β. A_1 consists of 2 closed intervals. Let A_2 be the set obtained from A_1 by deleting two open intervals centered at the middle of each closed interval of A_1 with a length β^2. A_2 consists of 2^2 closed intervals. This deleting process continues indefinitely. We denote by G the union of the intervals which were deleted and by D_α the remaining set. G is an open set with the Lebesgue measure

$$\beta + 2\beta^2 + 2^2\beta_3 = \beta(1 + 2\beta + (2\beta)^2 + \cdots) = \frac{\beta}{1-2\beta} = 1 - \alpha.$$

Thus, D_α is a Lebesgue measurable set with the Lebesgue measure α. The set D_α is a nowhere dense set. When $\alpha = 0$, one has $\beta = \frac{1}{3}$ and D_α is the Cantor set. However, since int $D_\alpha = \emptyset$, so it is nonrobust.

In the above process, after deleting each open interval, we add two B_0-B_1 type sets. For instance, after deleting the open interval $\left(\frac{1}{2}-\frac{\beta}{2},\frac{1}{2}+\frac{\beta}{2}\right)$ from A_0, we add sets $B_0\left(\frac{1}{2}-\frac{\beta}{2},\frac{1}{2}\right)$ and $B_1\left(\frac{1}{2},\frac{1}{2}+\frac{\beta}{2}\right)$. In general, after deleting 2^{k-1} open intervals of the form $\left(p-\frac{\beta^k}{2},p+\frac{\beta^k}{2}\right)$ from A_{k-1}, we add two sets $B_0\left(p-\frac{\beta^k}{2},p\right)$ and $B_1\left(p,p+\frac{\beta^k}{2}\right)$, where p is the middle of such deleted open interval. We denote by B the union of these B_0-B_1 type sets. Each B_0-B_1 type set is a union of countably many open intervals. Thus, the set B is also open. Let

$$D = D_\alpha \cup B. \tag{19}$$

We have int $D = B$; each point of B is a robust point of D. Furthermore, if $x \in D_\alpha$, then the intersection of each neighborhood $(x-\delta, x+\delta)$ of x with $G = [0,1]\setminus D_\alpha$ is nonempty; also nonempty is $B =$ int D. Hence, x is also a robust point of D. These imply that D is a robust set. Note that each point in D_α is a singular robust point of D. We have proved the following theorem.

THEOREM 5.1. *There is a robust set $D = D_\alpha \cup B$ in the interval [0,1], where B is open and D_α is nowhere dense set. Each point of D_α is a singular robust point of D. The Lebesgue measure of the set D is α which can be close to 1.*

Let

$$f(x) = \begin{cases} 0, & \text{for } x \in G = [0,1]\setminus D_\alpha, \\ 1, & \text{for } x \in D_\alpha. \end{cases}$$

Since G is open, it follows from Proposition 2.3 that f is a robust function; and in fact, f is continuous at each point of G. For each point $x \in D_\alpha$, the intersection of G and each neighborhood $(x-\delta, x+\delta)$ of x are nonempty; the function value at each point of the intersection is zero. Thus, f is discontinuous at $x \in D_\alpha$ (with a discontinuity of second kind, by Theorem 3.1); this proves the following result.

THEOREM 5.2. *There is a robust function on the interval [0,1] such that the Lebesgue measure of its set of points of discontinuity approaches 1.*

In the construction of D, $D = D_\alpha \cup B$, the Lebesgue measure of D_α is α; we can take it to be positive. We know that each set with a positive Lebesgue measure contains a Lebesgue nonmeasurable subset. Denote by D_p the Lebesgue nonmeasurable subset of D_α. Then the set $D_p \cup B$ is also a robust set. Indeed, we have int $(D_p \cup B) = B$ and each point of B is an interior point of $D_p \cup B$, so such point is a robust one of $D_p \cup B$. Furthermore, let $x \in D_p$; the intersection of int $(D_p \cup B) = B$ with each neighborhood $(x-\delta, x+\delta)$ is nonempty, thus, x is also a robust point of $D_p \cup B$. Hence, $D_p \cup B$ is a robust set. However, the set $D_p \cup B$ is a Lebesgue nonmeasurable set since B is Lebesgue measurable, D_p is Lebesgue nonmeasurable and $D_p \cap B = \emptyset$. Thus, we have the following theorem.

THEOREM 5.3. *There is a Lebesgue nonmeasurable robust subset in the interval [0,1].*

Utilizing the above theorem, we can construct a Lebesgue nonmeasurable robust function as follows. Let

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1]\setminus (D_p \cup B), \\ 0, & \text{if } x \in D_p \cup B. \end{cases} \tag{20}$$

Since $D_p \cup B$ is a robust set, it follows from Proposition 2.3 that f is a robust function. However, if we take $0 < c \leq 1$, the set

$$F_c = \{x \in [0,1] \mid f(x) < c\} = D_p \cup B$$

is Lebesgue nonmeasurable. Thus, function (20) is Lebesgue nonmeasurable which implies the next theorem.

THEOREM 5.4. *There is a Lebesgue nonmeasurable robust function on the interval [0,1].*

REFERENCES

1. R. Baire, *Leçons sur les Fonctions Discontinues*, Gauthier-Villars, Paris, (1905).
2. E. Borel, *Leçons sur les Fonctions de Variables Réelles*, Gauthier-Villars, Paris, (1905).
3. M.H. Lebesgue, Sur l'approximation des fonctions, *Bulletin des Sciences Mathematiques*, novembre, (1898).
4. J.C. Oxtoby, *Measure and Category*, Second Edition, Springer-Verlag, New York (1980).
5. H.L. Royden, *Real Analysis*, Second Edition, Macmillan Publ. Co., New York, (1968).
6. Q. Zheng, Global optimization of a class of discontinuous functions, *Numerical Analysis, A Journal of Chinese Universities*, (in Chinese), 8 (1), 31–43, (1985).
7. Q. Zheng, Robust analysis and global minimization of a class of discontinuous functions (I), *Acta Mathematicae Applicatae Sinica* (English Series), 6(3), 205–223,(1990).
8. Q. Zheng, Robust analysis and global minimization of a class of discontinuous functions (II), *Acta Mathematicae Applicatae Sinica* (English Series), 6(4), 317–337, (1990).
9. Q. Zheng, Robust analysis and global optimization, *International J., Computers and Mathematics with Applications*, **21**(6/7) 17–24, (1991).
10. E. Galperin and Q. Zheng, *Global Solutions in Optimal Control and Games*, NP Research Publ., Montreal (1991).
11. E. Galperin and Q. Zheng, Integral global optimization in functional spaces with applications, (to appear).
12. E. Hewitt and K. Strumberg, *Real and Abstract Analysis*, Springer-Verlag, New York (1975).
13. R.G. Kuller, *Topics in Modern Analysis*, Prentice-Hall, Englewood Cliffs, NJ (1969).